Salient Assemblage Slide Presentation

Salient Assemblage Representation of Multidimensional, Recursive, Deforming Geometry

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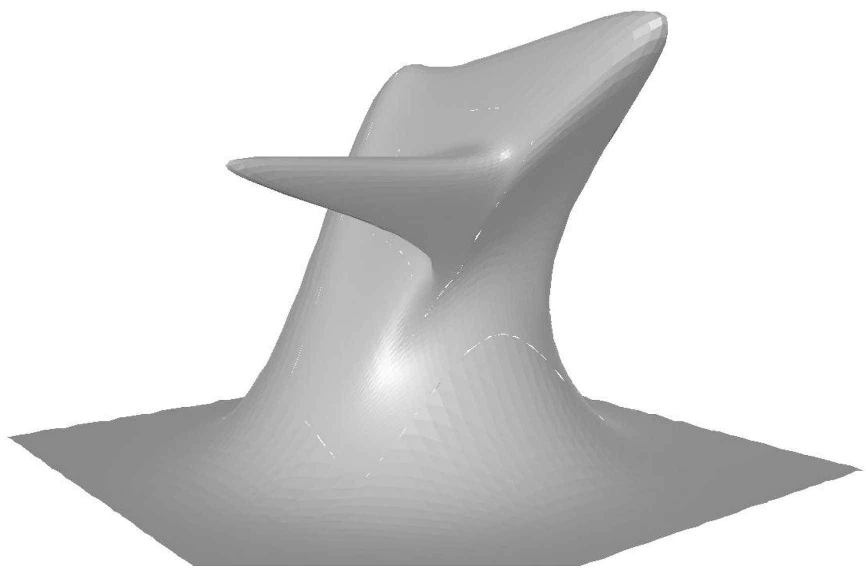
Prototypical Salient Assemblage



Assemblage constructed from 3 *salient* units.

Concise data storage (24 constants)

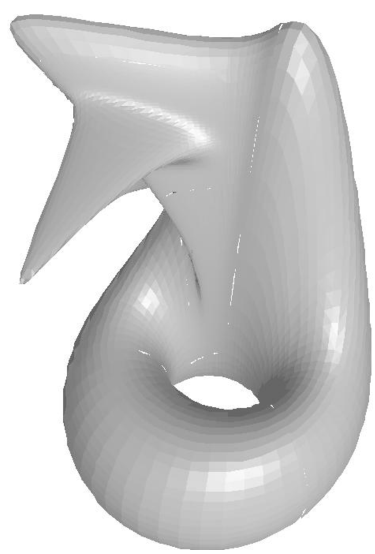
$$\begin{bmatrix} 3.0 & 0.0 & 1.0 & 0 & 0.0 & 0.0 & 1.0 & 0 \\ 2.0 & 0.0 & 0.5 & 0 & 0.0 & 0.5 & 0.5 & 0 \\ 2.0 & 0.5 & 0.2 & 0 & 0.0 & 0.5 & 0.2 & 0 \end{bmatrix}$$

Salient Assemblage Representation $y^r = f(x^i)$ Requirements

- Add, remove, reposition, deform salient units.
- Asymptotic C^∞ salient blending.
- Topologically invariant, homeomorphic with one parameter space.
- Local control of salient direction, shape, size, and volume, at least approximately.
- Recursive attachment rules, like alignment with principal directions.
- Applies to any dimension ($i = 1, 2, \ldots, n$).

Salient Assemblage is Topologically Invariant

Appended to Torus



Salient Assemblage Representation $y^r = f(x^i)$ Characteristics

- Constructive formulation, salient semiaxes form finite skeleton substructure.
- Concise data storage.
- One patch, thus no patch boundary, avoid geodesic cusp.
- Parameters usually have physical significance.
- Nowhere flat.
- Complicated algebraic expressions require computer.

Applications

- Parametric Systems (multidimensional)
  - Chemical reaction
  - Economy
  - Decision making
  - Geodesic determination
- Geometric Modeling (shape sensitive)
  - External fluid flow
  - Biological surface, deformation, growth
  - Telecommunicating complicated geometry using concise data storage

Key Issues

- What notation? Tensor notation for general curvilinear coordinate transformations.

- How to control salient direction, shape, and size, at least approximately.

- Account for parameter stretching and coordinate curve obliquity.

- Account for salient attachment in high-curvature regions.

- Efficiently compute complicated algebraic expressions.

Comparison with Other Mathematics
Frequently Asked Questions

- Why not conformal mapping? Powerful but too specialized—requires analytic mapping, preserves angle, limited to 2 dimensions, corresponds to *minimal surfaces*, a special class of manifolds. A salients has less restrictive C^∞ continuity and can be multidimensional.

- Why not 3D modeling, partition into small spline patches? Very complicated face, edge, and vertice relations in high dimensions. Patch boundaries complicate geodesic computation.

- Why not use Fourier Transform, making period arbitrarily large? Salient is more natural, not defined by a integral.

- Why is a salient a tensor-product surface? Efficient evaluation and partial derivatives, and easily extends to higher dimensions.

- Can a salient be a minimal surface? No. It has non-constant curvature. It is nowhere flat.

Comparison with Spline Representations

	Assemblage	Multi-patch Splines
primitive	salient	spline
formulation	function	discrete
recursive	yes	no
topology modeling	invariant	flexible
parameters	physical	arbitrary
patch coverage	large	small
patch boundary	C^∞	C^2
data storage	salient constants	control vertices

Both are parametric representations and are compatible.

Presentation Overview

1. Describe a salient.

2. Describe ExpHermite salient, a generalized Fourier series.

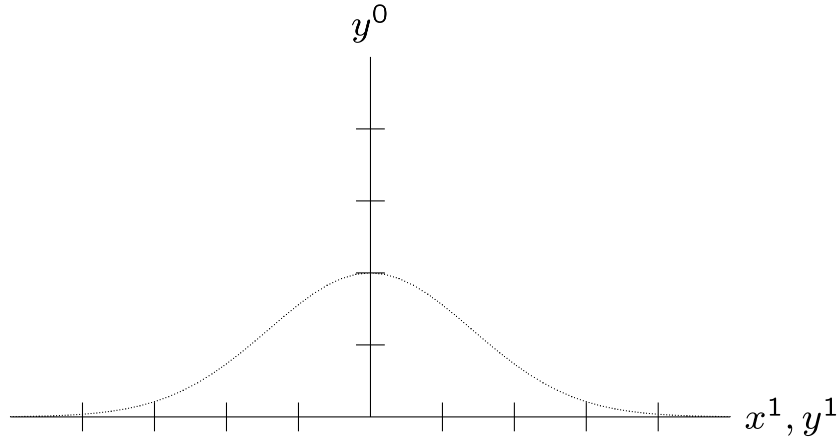
3. Describe salient attachment rules.

4. Derive parametric representation $y^r = f(x^i)$.

5. Apply differential geometry methods, e.g. geodesics.

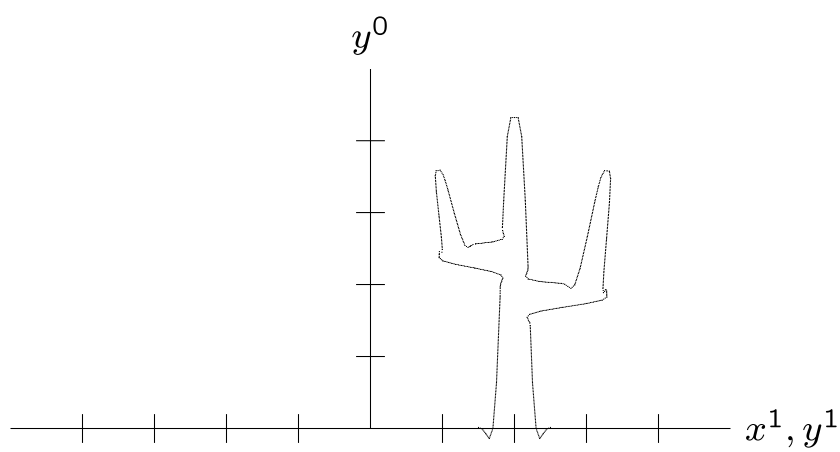
Definitions

Definition 1 *A* **salient** *is the mathematical representation of a distinguishable geometric part. It is a class C^∞ bounded function on $\mathbb{R}$ that, along with all its bounded derivatives, vanishes sufficiently far from one set of parametric arguments.*



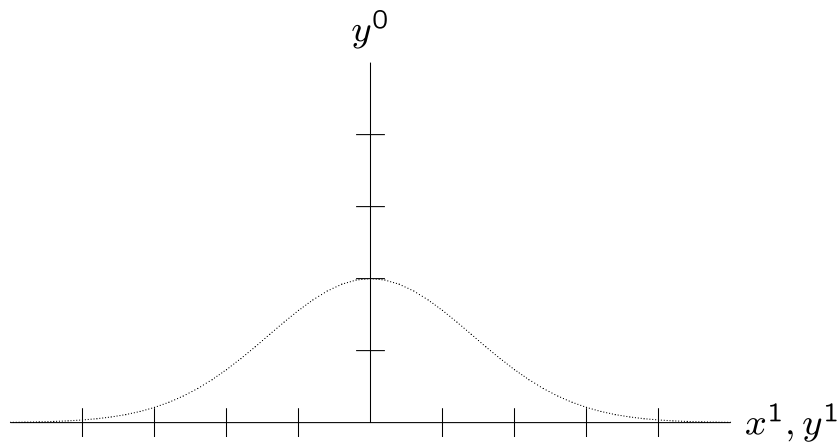
1D Salient

Definition 2 *An* **assemblage** *is a collection of attached salients.*

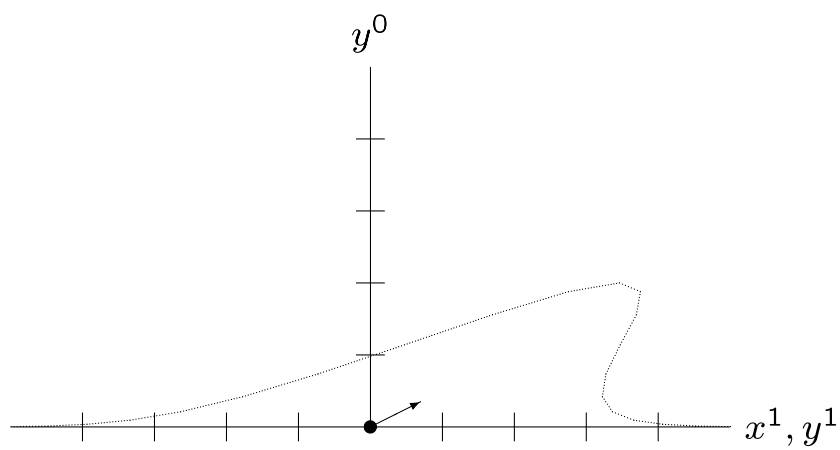


1D Assemblage

1D Salient



1D Salient

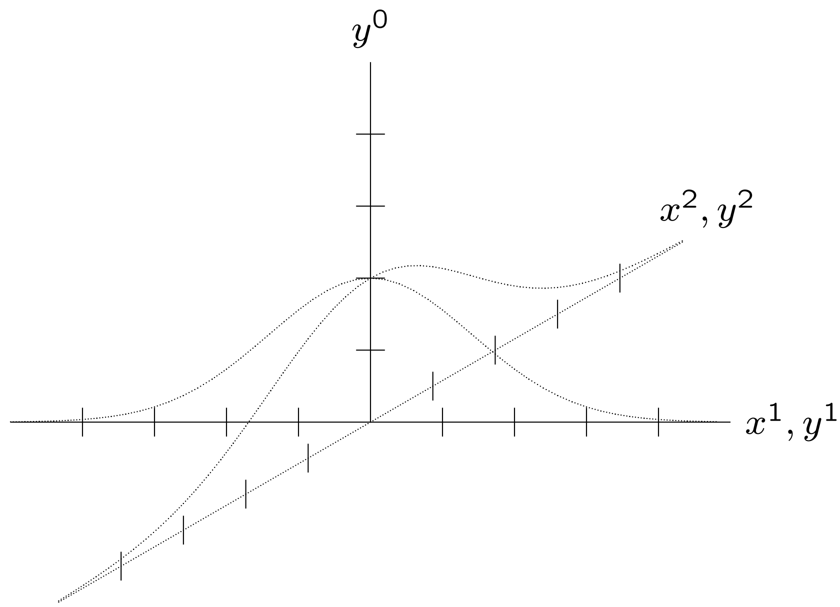


Main Semiaxis Direction (Dihedral)

$$
\begin{aligned}
y^0 &= \eta^0 \widehat{S}, \\
y^1 &= x^1 + \eta^1 \widehat{S},
\end{aligned}
$$

where η^r are direction cosines.

2D Salient



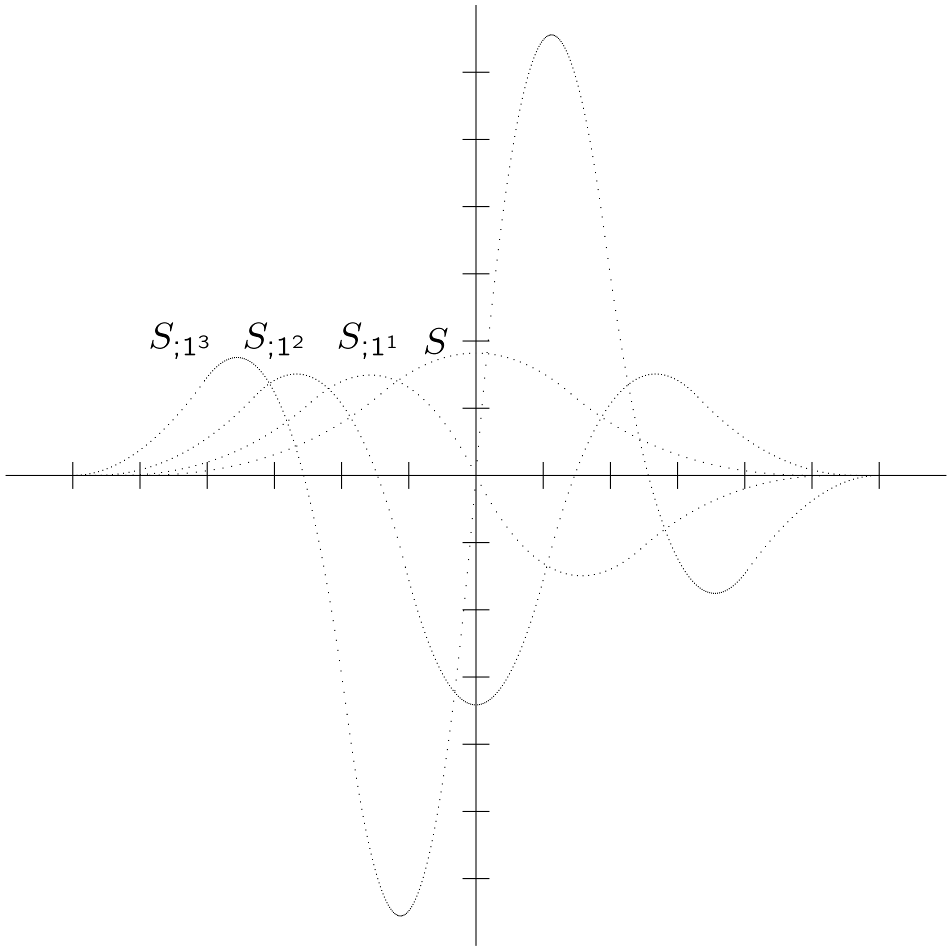
$$
\begin{aligned}
y^0 &= \eta^0 \widehat{S}, \\
y^1 &= x^1 + \eta^1 \widehat{S}, \\
y^2 &= x^2 + \eta^2 \widehat{S}.
\end{aligned}
$$

More concise notation for any dimension

$$y^r = \delta_i^r x^i + \eta^r \widehat{S},$$

where $r = 0, 1, \ldots, n; i = 1, 2, \ldots, n$
and $\eta^\rho \eta^\rho = 1$.

Salient Derivatives are Salients



1D Salient and Its First Three Derivatives

If 1D salient S and its derivatives are linearly independent, then linear combination

$$\begin{aligned}\widehat{S} &= c^0 S + c^1 S_{;1} + c^2 S_{;11} + \cdots + c^{n_h} S_{;1^{n_h}} \\ &= c^h S_{;1^h} \qquad \text{(sum on } h = 0, 1, \ldots, n_h).\end{aligned}$$

spans a wider collection of 1D salients.

2D Salient

A linear combination of a 2D salient $S(x^1, x^2)$ and its derivatives

$$\widehat{S} = c^{h_1 h_2} S_{;1^{h_1} 2^{h_2}}(x^1, x^2),$$

is also a 2D salient.

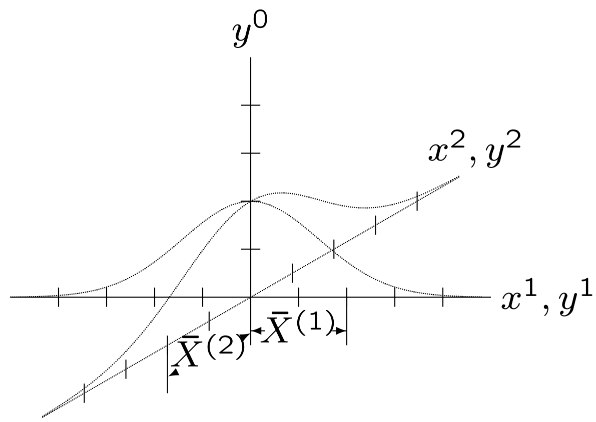
Consider only factorable S. Then $\widehat{S}$ is a *tensor-product surface*,

$$\begin{aligned} \widehat{S} &= \left(c_{(1)}^{h_1} S_{(1);1^{h_1}} \left(\bar{x}^1 \right) \right) \cdots \left(c_{(n)}^{h_n} S_{(n);n^{h_n}} \left(\bar{x}^n \right) \right), \\ &= \prod_j c_{(j)}^{h_j} S_{(j);j^{h_j}} \left(\bar{x}^j \right). \end{aligned}$$

Consequently,

$$y^r = \delta_i^r x^i + \eta^r \widehat{S} = \delta_i^r x^i + \eta^r \prod_j c_{(j)}^{h_j} S_{(j);j^{h_j}} \left(\bar{x}^j \right).$$

Salient Nomenclature



Although a salient is open and unbounded, ellipse nomenclature is useful.

Definition 3 *Salient* **origin***, denoted by $\dot{X}^i$, is the salient's local coordinate origin.*

Local curvilinear coordinates, centered on salient origin, are

$$\bar{x}^i \equiv x^i - \dot{X}^i.$$

Definition 4 *Salient* **main semiaxis** *is the line segment from salient origin in direction η^r.*

Definition 5 *Salient* **height** *is main semiaxis length.*

Definition 6 *Salient* **vertex** *is main semiaxis endpoint.*

Definition 7 *Salient $\bar{x}^j$-***semiaxis** *is the positive canonical coordinate $\bar{x}^j$ axis.*

Definition 8 *Salient semiaxis* **width** $\bar{X}^{(j)}$ *is the $\bar{x}^j$-semiaxis radial width at which salient height is $1/e$ times the main semiaxis height.*

Local Curvilinear to Canonical Coordinate Transformation

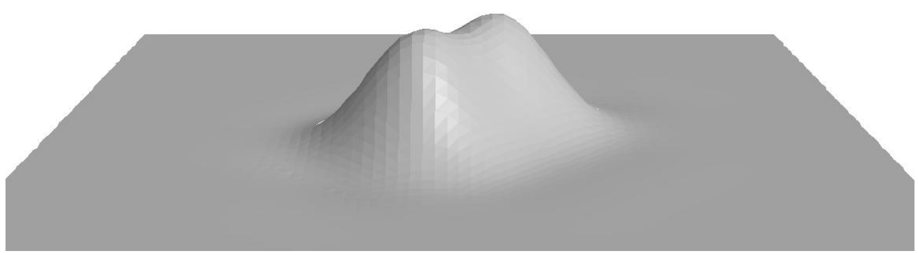
In two-dimensions, scaling and rotation transformations are

$$\left[\begin{array}{c}\bar{x}^1\\ \bar{x}^2\end{array}\right] = \left[\begin{array}{cc}1/\bar{X}^{(1)} & 0\\ 0 & 1/\bar{X}^{(2)}\end{array}\right]\left[\begin{array}{cc}\zeta_1^1 & \zeta_2^1\\ \\ \zeta_1^2 & \zeta_2^2\end{array}\right]\left[\begin{array}{c}\bar{x}^1\\ \bar{x}^2\end{array}\right].$$

In any dimension,

$$\bar{x}^j = \chi_\alpha^j \zeta_i^\alpha \bar{x}^i.$$

2D Salient (Tensor-Product Surface) with Two Shapes



Rectangle and Cone Approximations (5 terms)

In this case, salient semiaxes are rotated $\pi/4$ from rectangular axes.

Candidate Salient Functions

- $\exp\left(-x^2\right)$
- $2\exp(x)/(1+\exp(2x))$
- $\sin(ax)/x$
- Bessel function $J_0(x)$
- $J_1(x)/x$
- $\mathrm{sech}(x)$
- $1/(1+ax^2)$

Exponent-Salient Function

$$\exp\left(-\left((x^1)^2 + \cdots + (x^n)^2\right)\right) \equiv \exp\left(-x^i x^i\right).$$

continuous for parametric arguments but negligible sufficiently far from origin $(x^1, x^2, \ldots, x^n) = (0, 0, \ldots, 0)$.

Approximate values are:

x^1	$\exp\left(-(x^1)^2\right)$
0	1
1	0.36788
2	0.01831
3	1.23410×10^{-4}
4	1.12535×10^{-7}
5	1.38879×10^{-11}

Hermite Polynomials

Exponent-salient function has derivatives of all orders

$$\frac{d^h}{dx^h} \exp\left(-x^2\right) = \exp\left(-x^2\right) H_h(x) .$$

Definition 9 Hermite polynomials *are*

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= -2x, \\ H_{h+1}(x) &= -2\left(xH_h(x) + hH_{h-1}(x)\right) . \end{aligned}$$

First few Hermite Polynomials

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= -2x, \\ H_2(x) &= -2 + 4x^2, \\ H_3(x) &= 12x - 8x^3, \\ H_4(x) &= 12 - 48x^2 + 16x^4, \\ H_5(x) &= -120x + 160x^3 - 32x^5. \end{aligned}$$

ExpHermite Series

Hermite polynomial products, weighted by $\exp\left(-x^2\right)$, are orthogonal,

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right) H_h(x)\, H_\theta(x)\, dx = \begin{cases} 0 & \text{if } h \neq \theta \\ 2^h h! \sqrt{\pi} & \text{if } h = \theta. \end{cases}$$

Expand given salient function as an *ExpHermite series*

$$f(x) = \exp\left(-x^2\right) c^h H_h(x) \qquad \text{(sum on } h = 0, 1, \ldots, n_h\text{)}.$$

where c^h are *ExpHermite coefficients* and the ExpHermite series is a *generalized Fourier series*. To find c^h, multiply both sides by $H_\theta(x)$,

$$f(x) H_\theta(x) = \exp\left(-x^2\right) c^h H_h(x)\, H_\theta(x)\,.$$

Integrating both sides gives

$$\int_{-\infty}^{\infty} f(x) H_\theta(x)\, dx = c^h \int_{-\infty}^{\infty} \exp\left(-x^2\right) H_h(x)\, H_\theta(x)\, dx.$$

Because of orthogonality, for any particular h,

$$\int_{-\infty}^{\infty} f(x) H_h(x)\, dx = c^h \int_{-\infty}^{\infty} \exp\left(-x^2\right) \left(H_h(x)\right)^2 dx.$$

From first equation above,

$$c^h = \frac{1}{2^h h! \sqrt{\pi}} \int_{-\infty}^{\infty} f(x) H_h(x)\, dx.$$

ExpHermite Coefficients for Special Shapes

Using

$$c^h = \frac{1}{2^h h! \sqrt{\pi}} \int_{-\infty}^{\infty} f(x) H_h(x)\ dx,$$

determine coefficients:

	$f(x)$	c^0	c^2	c^4	c^6
exponent	$\exp\left(-x^2\right)$	1	0	0	0
rectangle	1	$\frac{2}{\sqrt{\pi}}$	$\frac{-1}{6\sqrt{\pi}}$	$\frac{-1}{240\sqrt{\pi}}$	$\frac{29}{20160\sqrt{\pi}}$
cone	$1-\|x\|$	$\frac{1}{\sqrt{\pi}}$	$\frac{-1}{6\sqrt{\pi}}$	$\frac{19}{1440\sqrt{\pi}}$	$\frac{-13}{20160\sqrt{\pi}}$
parabola	$1-x^2$	$\frac{4}{3\sqrt{\pi}}$	$\frac{-1}{5\sqrt{\pi}}$	$\frac{11}{840\sqrt{\pi}}$	$\frac{-37}{90720\sqrt{\pi}}$
semicircle	$\sqrt{1-x^2}$	$\frac{\sqrt{\pi}}{2}$	$\frac{-\sqrt{\pi}}{16}$	$\frac{\sqrt{\pi}}{384}$	$\frac{\sqrt{\pi}}{18432}$

These shapes are even functions with unit height and unit semiaxis width.

Approximation by ExpHermite Series

Rectangle

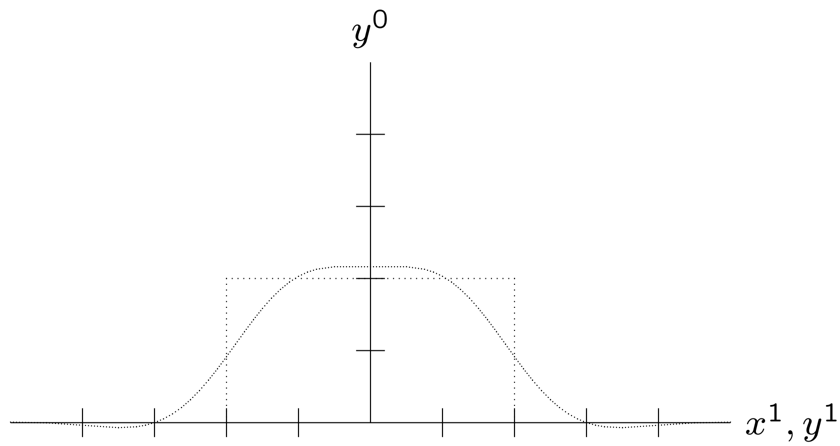
$$f(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{if } x > 1, \end{cases}$$

is approximated by

$$f(x) \approx \frac{\exp\left(-x^2\right)}{\sqrt{\pi}} \left(2 - \frac{1}{6}H_2(x) - \frac{1}{240}H_4(x) + \frac{29}{20160}H_6(x) - \frac{67}{580608}H_8(x)\right).$$

To compute, transform to power series

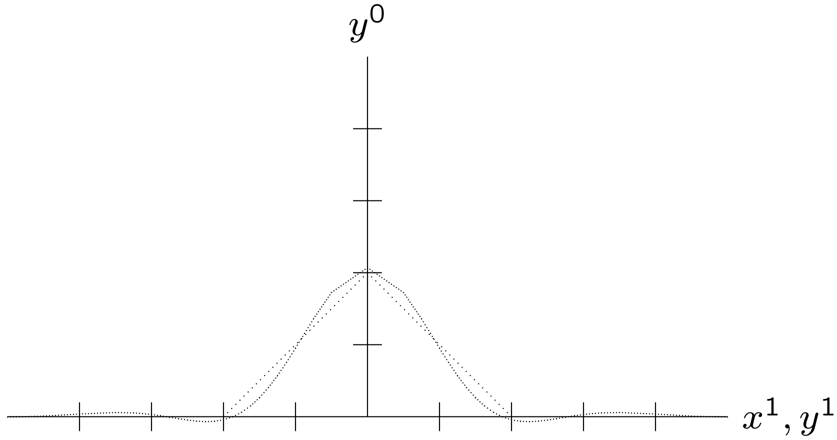
$$f(x) \approx \exp\left(-x^2\right)\left((((-0.01667x^2 + 0.28528)x^2 - 1.30219)x^2 + 1.19607)x^2 + 1.08147\right).$$



Approximation by ExpHermite Series

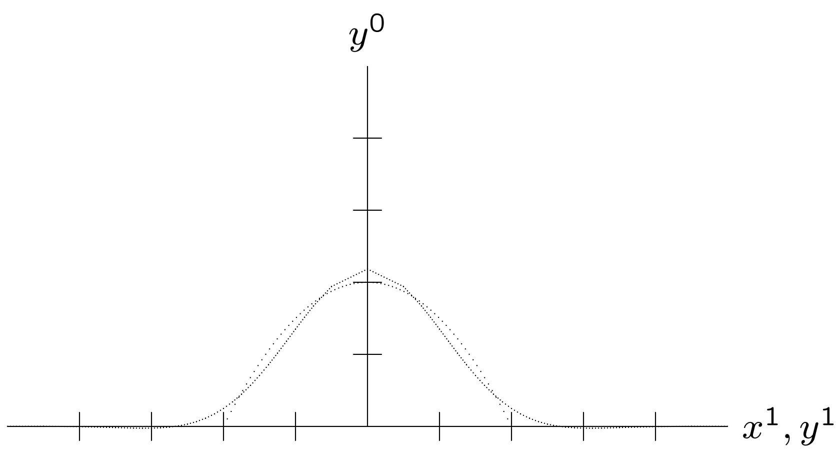
Cone approximation (5 terms)

$$f(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1 - |x| & \text{if } -1 \leq x \leq 1 \\ 0 & \text{if } x > 1. \end{cases}$$

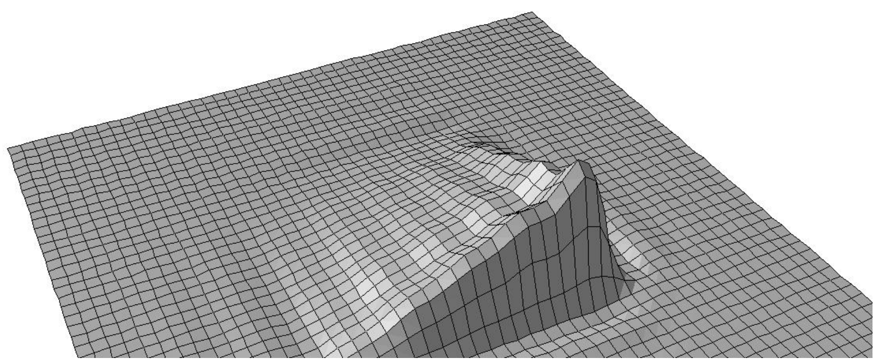


Parabola approximation (5 terms)

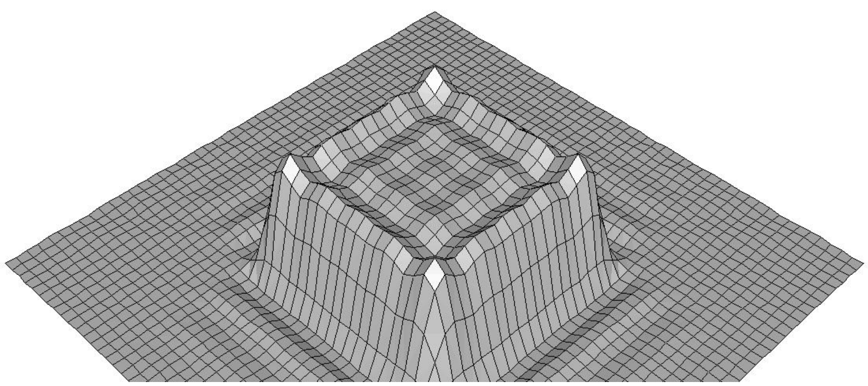
$$f(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1 - x^2 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{if } x > 1. \end{cases}$$



2D Ramp Approximation by Tensor-Product of ExpHermite Series



2D Rectangle Approximation by Tensor-Product of ExpHermite Series



ExpHermite Series Successive Approximations Change Shape but not Volume

Since

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right)\, dx = \sqrt{\pi},$$

and for $h > 0$,

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right) H_h(x)\, dx = 0,$$

then volume V under approximating surface is

$$\begin{aligned} V &= \int_{\mathcal{X}} \exp\left(-\bar{x}^\gamma \bar{x}^\gamma\right) \prod_j c_{(j)}^{h_j} H_{h_j}\left(\bar{x}^j\right)\, dx^1 \cdots dx^n, \\ &= \left(\prod_j c_{(j)}^0\right) \int_{\bar{\mathcal{X}}} \exp\left(-\bar{x}^\gamma \bar{x}^\gamma\right)\, d\bar{x}^1 \cdots d\bar{x}^n, \\ &= \pi^{n/2} \prod_j c_{(j)}^0. \end{aligned}$$

Assemblage Definitions

Definition 10 *An* **assemblage** *is a collection of attached salients.*

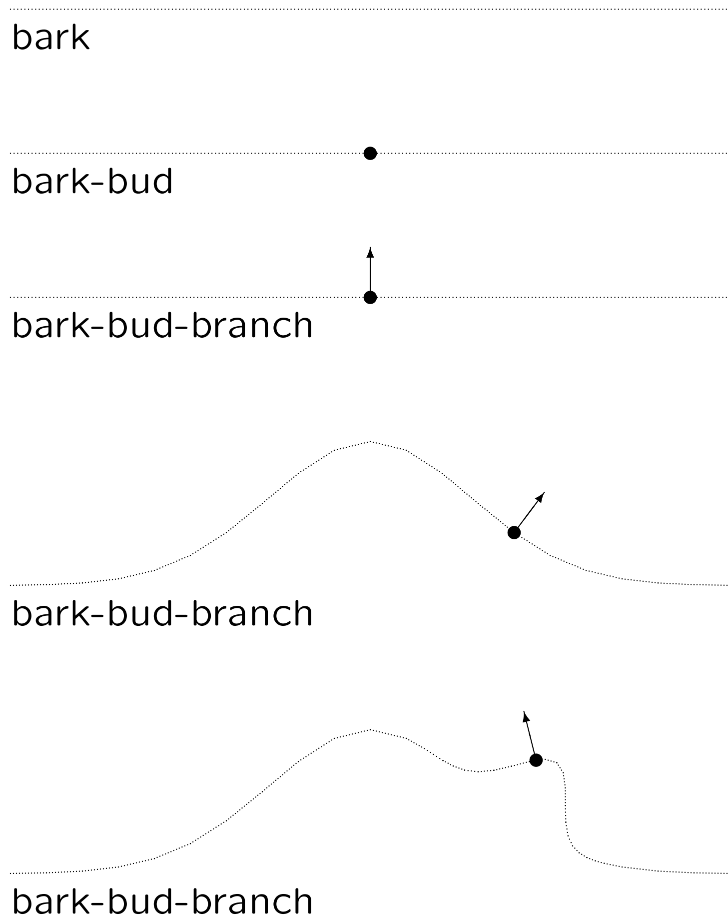
Definition 11 *A salient's* **parent** *is the assemblage to which it is attached.*

Definition 12 *A salient is a* **child** *to its parent.*

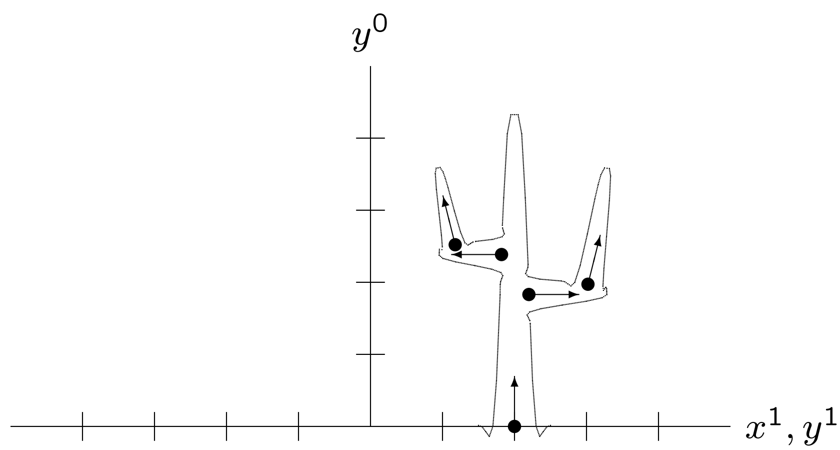
Definition 13 *A child's* **bud** *is the point* $\dot{Y}^r = y^r(\dot{X}^i)$*, located on the parent.*

Definition 14 *A child's* **dihedral** *is the minimum angle its main semiaxis forms with the parent's tangent plane at the bud.*

Salient Attachment by Vector Addition



Each salient depends on all its parents.

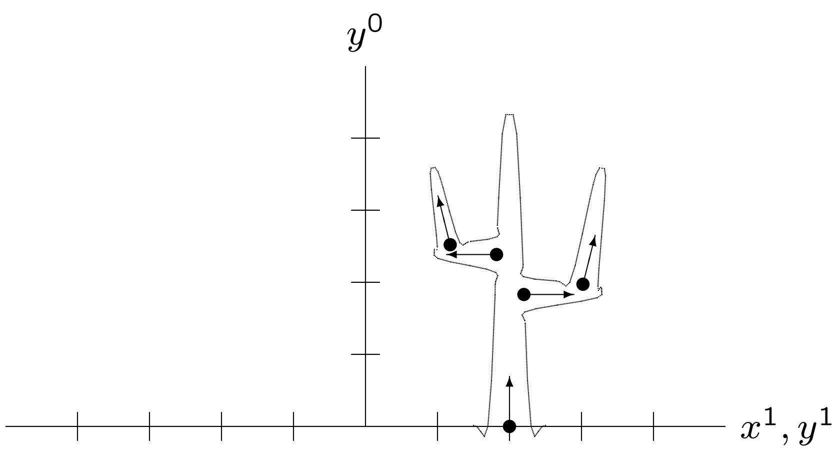


Salient Direction Cosine (Dihedral) Rule

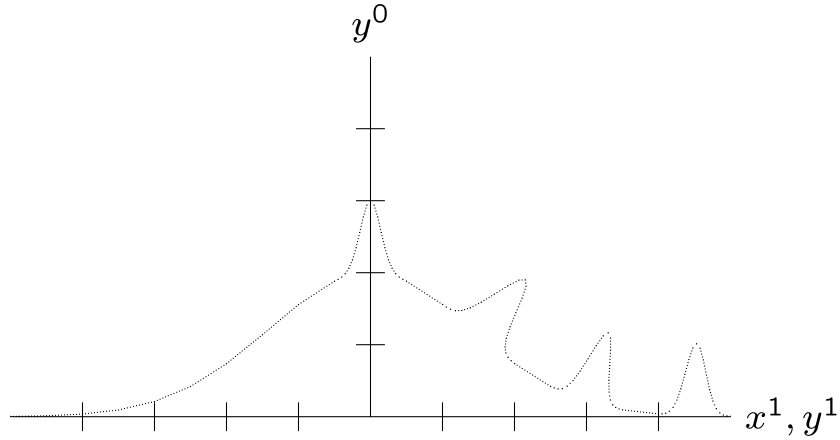
The mth salient main semiaxis can have any direction η^r_m, but usually is either:

- parent's unique normal vector,
- *branch angle*, coplanar with parent's positive main semiaxis,
- fixed angle to rectangular axes y^r.

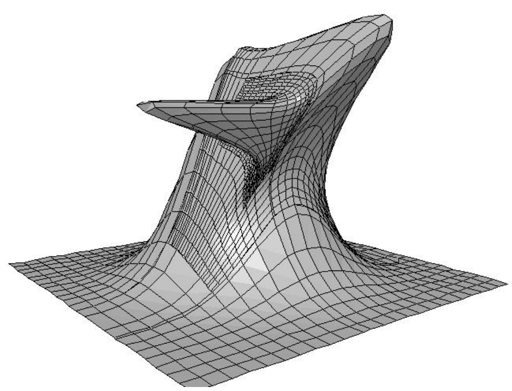
Rule can be an inherited.



Parameter Stretching and Coordinate Curve Obliquity



Child Salients Affected by Parameter Stretching



Coordinate Curve Obliquity

Arc-Length and Oblique Coordinate Transformations

Given by metric tensor g_{ij} at salient origin.

Arc-length coordinate transformation (2D)

$$[\lambda_i^\varepsilon] = \begin{bmatrix} \sqrt{g_{11}} & 0 \\ 0 & \sqrt{g_{22}} \end{bmatrix}.$$

Oblique coordinate transformation (2D)

$$[\omega_\varepsilon^\gamma] = \begin{bmatrix} 1 & \frac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}} \\ & \\ & \\ 0 & \sqrt{1 - \frac{(g_{12})^2}{g_{11}g_{22}}} \end{bmatrix},$$

from Gramm-Schmidt orthonormalization.

Semiaxis Alignment Coordinate Transformation

Semiaxes are aligned with either:

- principal directions at bud, eigenvectors of

$$[b_{i\alpha}]\,[x^\alpha] = \kappa\,[g_{i\alpha}]\,[x^\alpha]\,.$$

- *branch angle* direction, in the normal section that is parallel to the parent's positive main semiaxis,

- fixed direction relative to rectangular axes y^r,

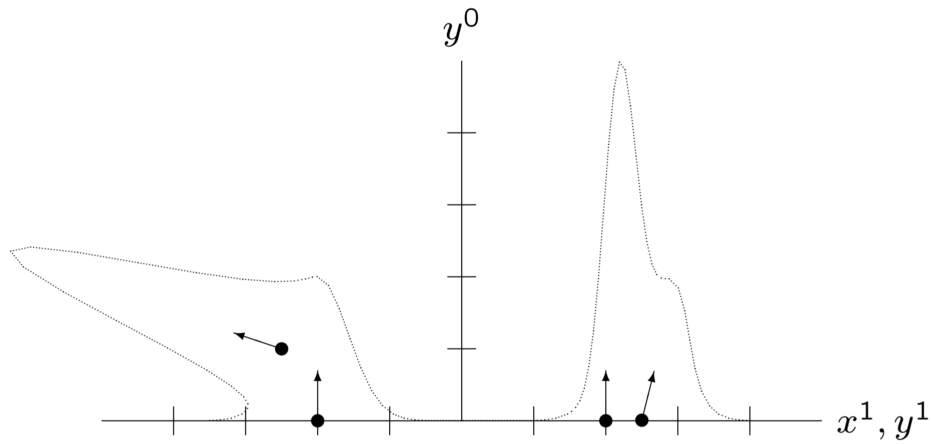
- one coordinate curve tangent vector.

Rule can be an inherited.

Salient Addition

Salient addition is closed.

Addition of a coupled salient is non-commutative and non-associative.

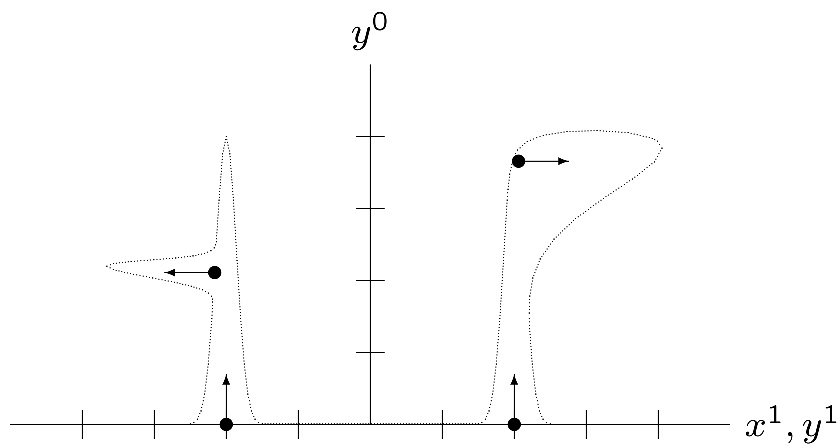


Non-commutative Salient Addition

Salient Attachment in High-Curvature Regions

Definition 15 *If child salient is smaller, the interaction is* **hierarchical** *or* **tree-like**.

Definition 16 *If child salient is approximately the same size or larger, the interaction is* **tumor-like**, *or if flattened* **anvil-like**.



Tree-like and Tumor-like Salient Interaction

Two widely separated salients m_1 and m_2 are approximately orthogonal,

$$\int_{\mathbb{R}^n} |\widehat{S}^{m_1}\,\widehat{S}^{m_2}|\,dx^1\,dx^2 \cdots dx^n \approx 0.$$

Salient Assemblage Representation

Overall coordinate transformation

$$\Upsilon^{j}_{(m)i} \equiv \chi^{j}_{(m)\alpha}\zeta^{\alpha}_{(m)\beta}\varsigma^{\beta}_{(m)\gamma}\omega^{\gamma}_{(m)\varepsilon}\lambda^{\varepsilon}_{(m)i}.$$

Canonical coordinates

$$\bar{x}^{j}_{(m)} = \Upsilon^{j}_{(m)i}\left(x^{i} - \dot{X}^{i}_{(m)}\right).$$

Parametric representation

$$\begin{aligned} y^{r} &= \delta^{r}_{i}x^{i} + \eta^{r}_{m}\widehat{S}^{m}, \\ &= \delta^{r}_{i}x^{i} + \eta^{r}_{m}\prod_{j} c^{h_j}_{(mj)} S^{m}_{(j);j^{h_j}}\left(\bar{x}^{j}_{(m)}\right). \end{aligned}$$

First partial derivative

$$\begin{aligned} y^{r}_{,k} &= \delta^{r}_{k} + \eta^{r}_{m}\Upsilon^{\alpha}_{(m)k}\widehat{S}^{m}_{,\alpha}, \\ &= \delta^{r}_{k} + \eta^{r}_{m}\Upsilon^{\alpha}_{(m)k}\prod_{j} c^{h_j}_{(mj)} S^{m}_{(j);j^{h_j+\delta^{\alpha}_{j}}}\left(\bar{x}^{j}_{(m)}\right). \end{aligned}$$

Second partial derivative

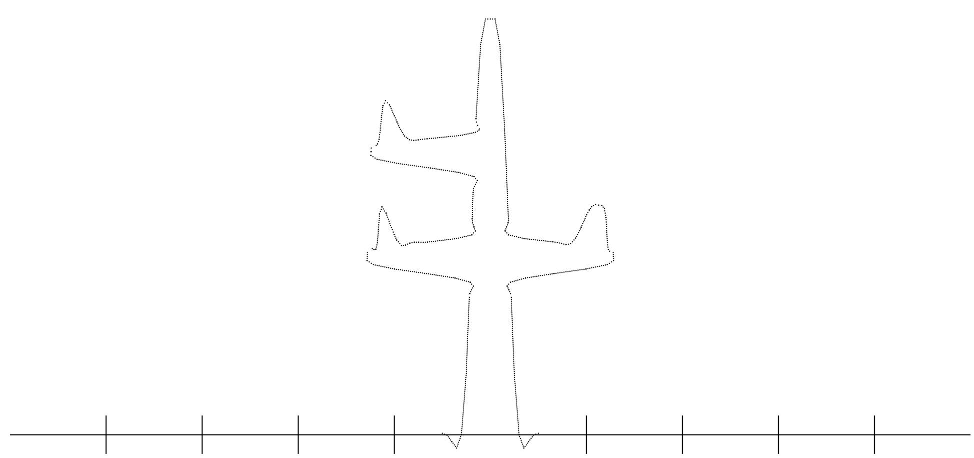
$$\begin{aligned} y^{r}_{,kl} &= \eta^{r}_{m}\Upsilon^{\alpha}_{(m)k}\Upsilon^{\beta}_{(m)l}\widehat{S}^{m}_{,\alpha\beta}, \\ &= \eta^{r}_{m}\Upsilon^{\alpha}_{(m)k}\Upsilon^{\beta}_{(m)l}\prod_{j} c^{h_j}_{(mj)} S^{m}_{(j);j^{h_j+\delta^{\alpha}_{j}+\delta^{\beta}_{j}}}\left(\bar{x}^{j}_{(m)}\right). \end{aligned}$$

1D ExpHermite Assemblage

ExpHermite salients in the form

$$\widehat{S}^m = \mathsf{exp}^m \left(-(\bar{x}^1_{(m)})^2 \right) c^h_{(m)} H_h \left(\bar{x}^1_{(m)} \right).$$

combine to form assemblage like



Cone, Parabola, and Rectangle in Tree

ExpHermite Salient Assemblage Representation

Parametric representation

$$y^r = \delta^r_i x^i + \eta^r_m \,\mathsf{exp}^m \left(-\bar{x}^\gamma_{(m)} \bar{x}^\gamma_{(m)} \right) \prod_j c^{h_j}_{(mj)} H_{h_j} \left(\bar{x}^j_{(m)} \right).$$

First partial derivative

$$y^r_{,k} = \delta^r_k + \eta^r_m \Upsilon^\alpha_{(m)k} \,\mathsf{exp}^m \left(-\bar{x}^\gamma_{(m)} \bar{x}^\gamma_{(m)} \right) \prod_j c^{h_j}_{(mj)} H_{h_j+\delta^\alpha_j} \left(\bar{x}^j_{(m)} \right).$$

Second partial derivative

$$y^r_{,kl} = \eta^r_m \Upsilon^\alpha_{(m)k} \Upsilon^\beta_{(m)l} \,\mathsf{exp}^m \left(-\bar{x}^\gamma_{(m)} \bar{x}^\gamma_{(m)} \right) \prod_j c^{h_j}_{(mj)} H_{h_j+\delta^\alpha_j+\delta^\beta_j} \left(\bar{x}^j_{(m)} \right).$$

Global Cylindrical Coordinates

(ρ, θ, z) to rectangular y^r

$$
\begin{aligned}
y^0 &= z, \\
y^1 &= \rho \cos\theta, \\
y^2 &= \rho \sin\theta.
\end{aligned}
$$

Inverse transformation

$$
\begin{aligned}
z &= y^0, \\
\rho &= \sqrt{(y^1)^2 + (y^2)^2}, \\
\theta &= \tan^{-1}\left(y^2/y^1\right).
\end{aligned}
$$

Global cylinder $\rho = R$ is

$$
\begin{aligned}
x^1 &= R\sin\theta, \\
x^2 &= z.
\end{aligned}
$$

Global point $(\dot{\Theta}, \dot{Z})$ becomes m_0 salient origin

$$
\begin{aligned}
\dot{X}^1_{(0)} &= R\sin\dot{\Theta}_{(0)}, \\
\dot{X}^2_{(0)} &= \dot{Z}_{(0)}.
\end{aligned}
$$

Global Spherical-Polar Coordinates

(ρ, ϕ, θ) to rectangular y^r

$$\begin{aligned} y^0 &= \rho \cos\phi, \\ y^1 &= \rho \sin\phi \cos\theta, \\ y^2 &= \rho \sin\phi \sin\theta. \end{aligned}$$

Inverse transformation

$$\begin{aligned} \rho &= \sqrt{(y^0)^2 + (y^1)^2 + (y^2)^2}, \\ \phi &= \tan^{-1}\left(\sqrt{(y^1)^2 + (y^2)^2}/y^0\right), \\ \theta &= \tan^{-1}\left(y^2/y^1\right). \end{aligned}$$

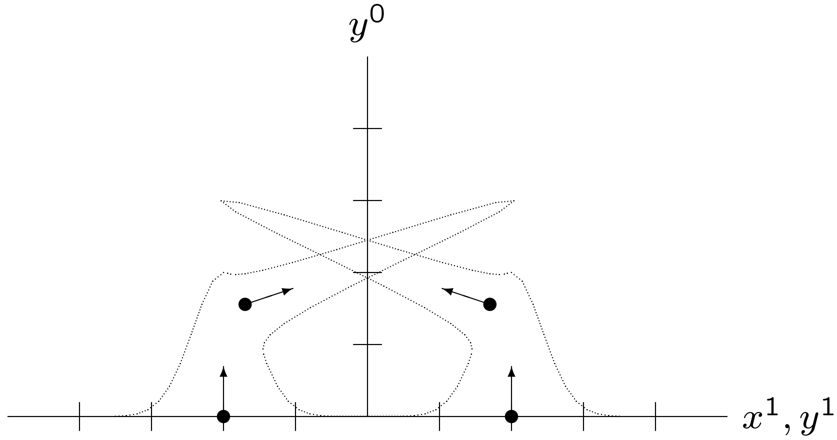
Global sphere $\rho = R$ is

$$\begin{aligned} x^1 &= R \sin\phi \cos\theta, \\ x^2 &= R \sin\phi \sin\theta, \end{aligned}$$

Global point $(\dot\Theta, \dot\Phi)$ becomes m_0 salient origin

$$\begin{aligned} \dot X^1_{(0)} &= R \sin\dot\Phi_{(0)} \cos\dot\Theta_{(0)}, \\ \dot X^2_{(0)} &= R \sin\dot\Phi_{(0)} \sin\dot\Theta_{(0)}. \end{aligned}$$

Assemblage Self-Intersection



Position vectors y^r of main semiaxes:

$$y^{r(m_1)} = \dot{Y}^r_{(m_1)} + t_1 \eta^r_{(m_1)},$$
$$y^{r(m_2)} = \dot{Y}^r_{(m_2)} + t_2 \eta^r_{(m_2)},$$

where $\dot{Y}^r_{(m_1)}$ and $\dot{Y}^r_{(m_2)}$ are buds, and t_1 and t_2 are scalar real parameters. Perpendicular connecting vector

$$\left(y^{r(m_1)} - y^{r(m_2)}\right) \eta_{r(m_1)} = 0,$$
$$\left(y^{r(m_1)} - y^{r(m_2)}\right) \eta_{r(m_2)} = 0.$$

So

$$\begin{bmatrix} \eta^r_{(m_1)}\eta_{r(m_1)} & -\eta^r_{(m_2)}\eta_{r(m_1)} \\ \eta^r_{(m_1)}\eta_{r(m_2)} & -\eta^r_{(m_2)}\eta_{r(m_2)} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} =$$

$$\begin{bmatrix} -(\dot{Y}^r_{(m_1)} - \dot{Y}^r_{(m_2)})\eta_{r(m_1)} \\ -(\dot{Y}^r_{(m_1)} - \dot{Y}^r_{(m_2)})\eta_{r(m_2)} \end{bmatrix}$$

Concise Data Storage

Salient-constant array for 2D assemblage

$$\begin{bmatrix} c_{(0)} & \dot{X}^1_{(0)} & \bar{X}^{(1)}_{(0)} & \mathsf{shape}^1_{(0)} & \zeta_{(0)} & \dot{X}^2_{(0)} & \bar{X}^{(2)}_{(0)} & \mathsf{shape}^2_{(0)} \\ c_{(1)} & \dot{X}^1_{(1)} & \bar{X}^{(1)}_{(1)} & \mathsf{shape}^1_{(1)} & \zeta_{(1)} & \dot{X}^2_{(1)} & \bar{X}^{(2)}_{(1)} & \mathsf{shape}^2_{(1)} \\ c_{(2)} & \dot{X}^1_{(2)} & \bar{X}^{(1)}_{(2)} & \mathsf{shape}^1_{(2)} & \zeta_{(2)} & \dot{X}^2_{(2)} & \bar{X}^{(2)}_{(2)} & \mathsf{shape}^2_{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

One row for each salient.

Facilitate telecommunicating a complicated geometry.

Efficient Computation

Assemblage

- Predict negligible terms from parameter values.
- Univariate factors in tensor-product.
- If recursive formula exists and is more efficient, use it.
- For non-deforming, precompute assemblage constants.
- For non-deforming, precompute coupling matrix.
- For partial derivatives, reuse previously computed function evaluations and repeating chain-rule factors.

ExpHermite Assemblage

- Exponent function $\exp\left(-x^2\right)$ factors out, leaving efficient polynomial.
- Transform Hermite series to power series.
- If factor is even or odd function, half the terms are zero and can be bypassed.

Transform Hermite Series to Power Series

Hermite series has equivalent power series

$$d^{q_j}_{(mj)\delta^{\alpha}_{j}h_k} P_{q_j}\left(\bar{x}^{j}_{(m)}\right) = c^{h_j}_{(mj)} H_{h_j+\delta^{\alpha}_{j}h_k}\left(\bar{x}^{j}_{(m)}\right),$$

where

$$P_{q_j}\left(\bar{x}^{j}_{(m)}\right) \equiv \left(\bar{x}^{j}_{(m)}\right)^{q_j}.$$

Coefficients transform as

$$d^{q_j}_{(mj)h_k} = \varpi^{q_j}_{\vartheta}\,\delta^{\vartheta}_{\theta+h_k} c^{\theta}_{(mj)},$$

where

$$\left[\varpi^{q_j}_{\vartheta}\right] = \begin{bmatrix} 1 & 0 & -2 & 0 & 12 & \\ 0 & -2 & 0 & 12 & 0 & \\ 0 & 0 & 4 & 0 & -48 & \\ 0 & 0 & 0 & -8 & 0 & \\ 0 & 0 & 0 & 0 & 16 & \\ & & & & & \ddots \end{bmatrix}.$$

Multiplication $\delta^{\vartheta}_{\theta+h_k} c^{\theta}_{(mj)}$ is equivalent to a shift of array $\mathbf{c}$ elements.

Differential Geometry

Parametric representation and its first two partial derivatives

$$y^r, \;\; y^r_{,k}, \;\; y^r_{,kl}.$$

Jacobian matrix

$$J \equiv [J^r_i]_{(n+1)\times n} \equiv y^r_{,i}.$$

Base vectors are functions of position (curvilinear coordinates)

$$\mathbf{a}_i = \mathbf{y}_{,i} \equiv y^\rho_{,i}\mathbf{e}_\rho.$$

Metric tensor

$$g_{ij} \equiv \mathbf{a}_i\,\mathbf{a}_j.$$

Since y^r are rectangular coordinates with the *Euclidean metric*,

$$\left[g_{ij}\right] = J^T J = \left[y^\rho_{,i}\, y^\rho_{,j}\right].$$

Differential Geometry

A more general *Riemannian metric*, including non-Euclidean, is

$$\left[g_{ij}\right] = J^T G J.$$

If J is full rank, then there exists a g^{ij} such that

$$g^{i\alpha}\, g_{\alpha j} = \delta^i_j.$$

Metric tensor determinant

$$g \equiv |g_{ij}|.$$

For two dimensions, $g = g_{11} g_{22} - (g_{12})^2$.

Cosine between x^i and x^j-parametric curves

$$\cos\omega = g_{ij}/\sqrt{g_{ii}\, g_{jj}} \qquad \text{(no sum on } i, j\text{)}.$$

Invariant *First Fundamental Form*

$$I \equiv g_{ij} dx^i\, dx^j.$$

Differential Geometry

Christoffel symbols of the first kind

$$\Gamma_{ijk} \equiv \frac{1}{2}\left(g_{jk,i} + g_{ki,j} - g_{ij,k}\right).$$

With rectangular coordinates y^r and Euclidean metric

$$\Gamma_{ijk} = y^{\rho}_{,k}\, y^{\rho}_{,ij}.$$

Christoffel symbols of the second kind

$$\Gamma^{k}_{ij} \equiv g^{k\alpha}\Gamma_{ij\alpha}.$$

Riemannian tensor of the second kind

$$R^{i}_{jkl} \equiv \Gamma^{i}_{jl,k} - \Gamma^{i}_{jk,l} + \Gamma^{\alpha}_{jl}\Gamma^{i}_{\alpha k} - \Gamma^{\alpha}_{jk}\Gamma^{i}_{\alpha l}.$$

Riemannian tensor of the first kind

$$R_{ijkl} \equiv g_{i\alpha}R^{\alpha}_{jkl}.$$

Gaussian curvature on a two-dimensional surface

$$K = \frac{R_{1212}}{g_{11}g_{22} - (g_{12})^2}.$$

Differential Geometry

Surface curve, function of parameter t,

$$x^i \equiv x^i(t),$$

in ambient coordinates y^r, is composite function

$$y^r \equiv y^r(x^i(t)).$$

Curve's tangent is given by chain-rule

$$\frac{dy^r}{dt} = y^r_{,i}\frac{dx^i}{dt}.$$

Square of differential arc length is

$$(ds)^2 = g_{ij}\, dx^i\, dx^j.$$

Curve arc length is

$$s = \int_{x_0}^{x_1} \sqrt{g_{ij}\, dx^i\, dx^j} = \int_{t_0}^{t_1} \sqrt{g_{ij}\, \frac{dx^i}{dt}\, \frac{dx^j}{dt}}\, dt.$$

Two-dimensional surface area is

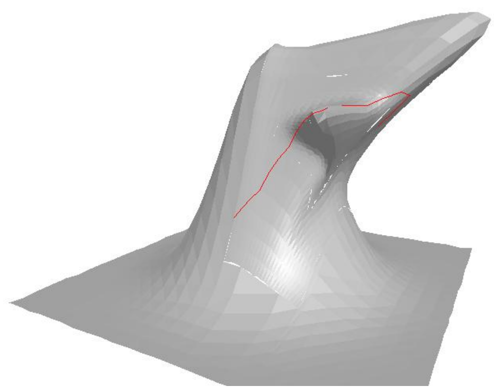
$$A = \int_{\mathcal{X}} \sqrt{g}\, dx^1\, dx^2.$$

Geodesic

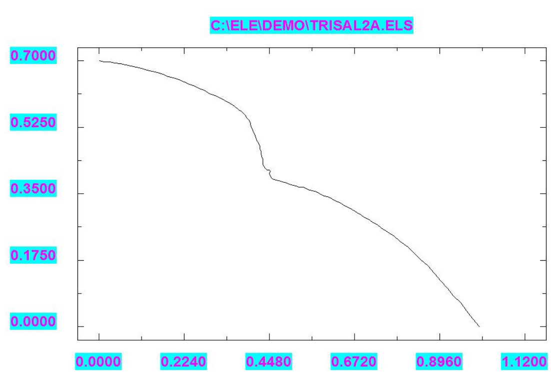
A *geodesic* $x^i(s)$ solves system of differential equations

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0.$$

Geodesic on surface (http://www.netlib.org/ode/geodesic/)



Geodesic in parameter space



Differential Geometry

Surface normal vector

$$N_r = \epsilon_{rst}\, y^s_{,1}\, y^t_{,2}.$$

Unit normal vector

$$n_r \equiv N_r/\sqrt{N_\rho\, N_\rho}.$$

Invariant *Second Fundamental Form*

$$II \equiv b_{ij} dx^i\, dx^j,$$

with coefficients from *curvature tensor*

$$b_{ij} \equiv y^\rho_{,ij} n_\rho.$$

For two dimensions,

$$b \equiv |b_{ij}| = b_{11} b_{22} - (b_{12})^2.$$

Gauss equation

$$y^r_{,ij} = \Gamma^\alpha_{ij}\, y^r_{,\alpha} + b_{ij} n^r.$$

Weingarten equation

$$n^r_{,i} = -g^{\alpha\beta} b_{i\alpha} y^r_{,\beta} = -b^\beta_i y^r_{,\beta}.$$

Tensor Applications

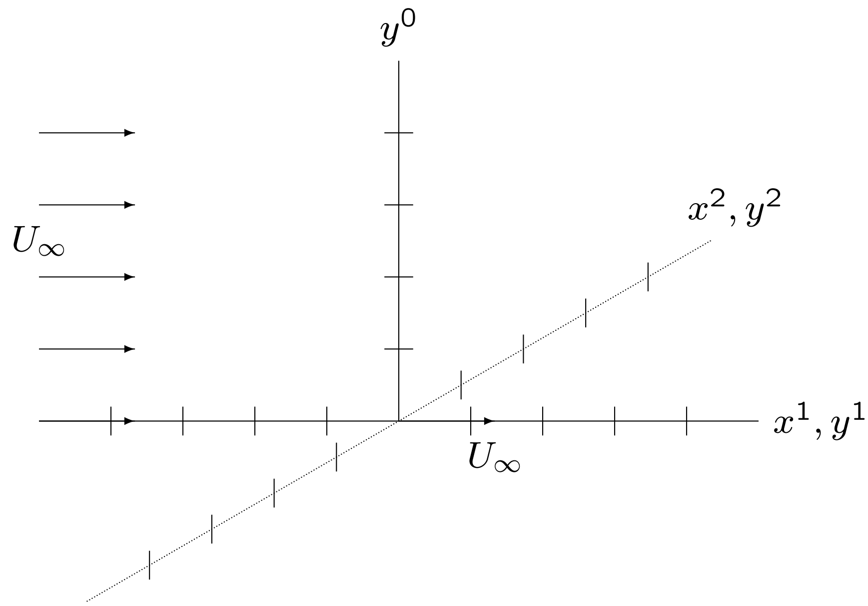
Newton's second law

$$\begin{aligned} F^r &= m\frac{dv^r}{dt}, \\ &= m\left(\frac{d^2y^r}{dt^2} + \Gamma^r_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right), \end{aligned}$$

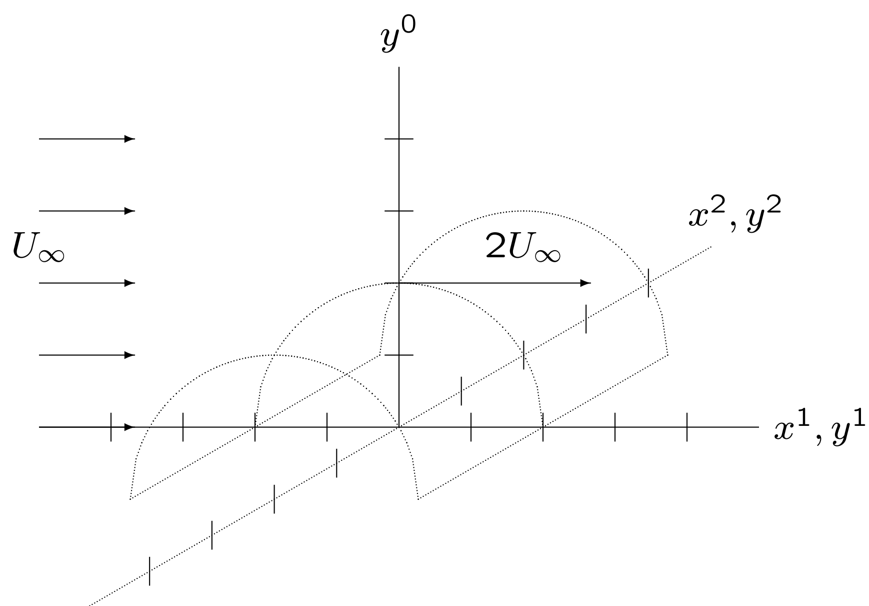
is valid in all coordinate systems.

Describes a force field on a curved surface, like the interface between two fluids.

Potential Flow Examples

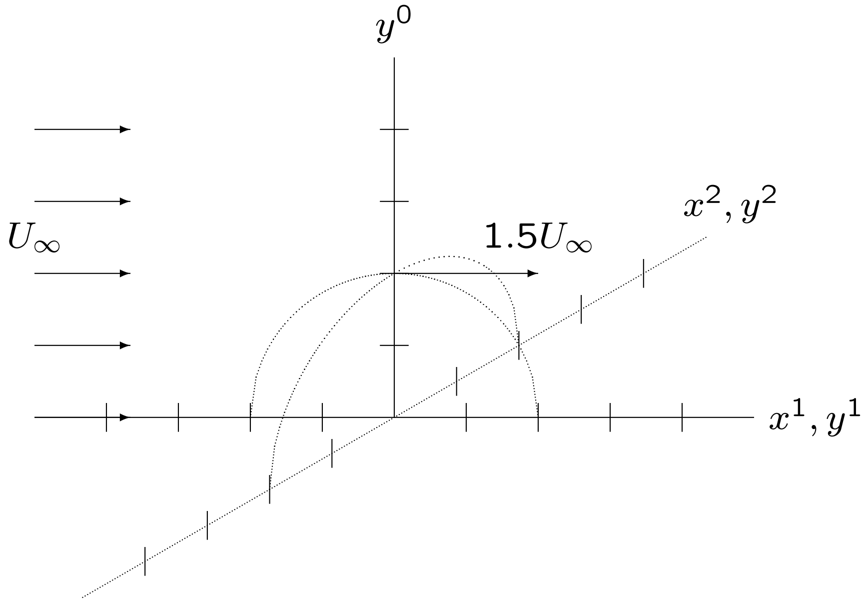


Potential Flow over Plane: $\phi = -U_\infty x^2$

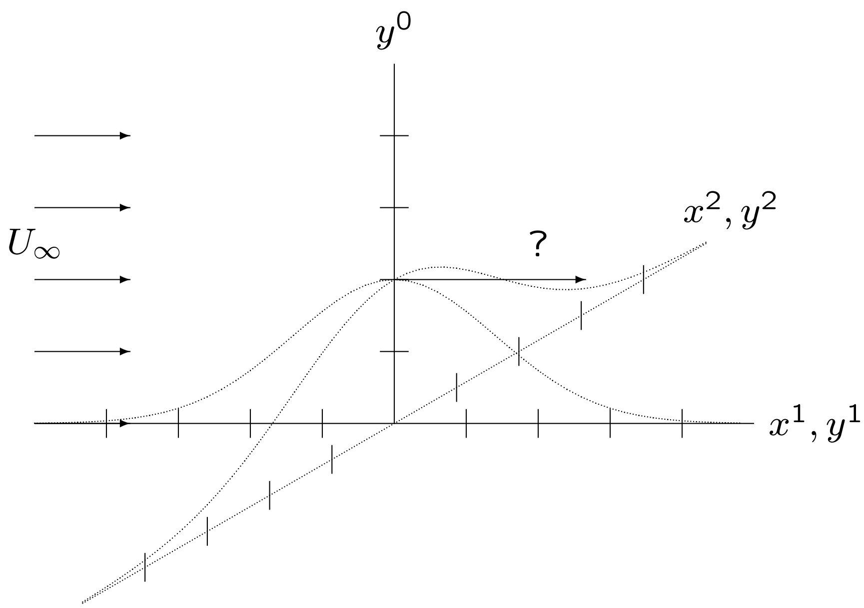


Potential Flow over Cylinder: $\phi = -U_\infty \left(1 + \frac{R^2}{r^2}\right) r \cos\theta$

Potential Flow Examples



Potential Flow over Sphere: $\phi = -U_\infty \left(1 + \frac{R^3}{2r^3}\right) r \cos\theta$



Potential Flow over Salient

Conclusions

Salient assemblage representation is important because:

- Complexity of many problems stems from representation of irregular or deforming geometry. An assemblage decomposes a geometric object into asymptotic blending salient units. It models a multidimensional parametric system.

- A linear combination of a 1D salient and its derivatives spans a wider collection of 1D salients.

- Salients are attached with recursive rules on dihedral and semiaxis alignment.

- An assemblage covers the surface of interest with one patch.

- An assemblage has concise data storage.

- It allows efficient computation.

ExpHermite assemblage representation has advantages:

- Built-in data fitting using ExpHermite series, a generalized Fourier series.

- Efficient polynomial computation.